ON THE EXACT CALCULATION OF OVERFLOW SYSTEMS

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ABSTRACT

Alternate routing systems with overflow facilities are widely used in long distance telephone networks. In such systems, calls can be switched via several routes. A very simple example of an alternate routing system, i.e. an overflow system, is shown in fig. 1. Calls which can not be switched via the first route (primary group) overflow to the final route (secondary group).

The topic of this paper is the exact calculation of such overflow systems.

The primary group as well as the secondary group of an overflow system can either be a full available group, or an ideal grading, or a non-ideal grading. Thus there are 9 possible types of overflow systems. An exact solution is given for all of these types of systems in case of Poisson input, i.e. an infinite number of traffic sources. (Exact solutions which are already known will only be briefly referred to.)

Section 1 deals with systems consisting of two non-ideal gradings. Section 2 is concerned with overflow systems in which one group is an ideal grading or a full available group. In section 3, systems with ideal gradings and full available groups are considered.

Finally, section 4 treats overflow systems with two full available groups and a finite number of traffic sources.

1. OVERFLOW SYSTEMS CONSISTING OF NON-IDEAL GRADINGS

In telephone networks, overflow systems with non-ideal gradings are used to a great extent. A very simple example of such a system is shown in fig. 2. The primary group is a grading with

Fig. 1: Simple example of an overflow system switched via the first route (primary group) overflow to the final route (secondary group).

The total loss $B_{tot}$ (or the traffic $R_2$ overflowing behind the secondary group, resp.) can be calculated according to the same method. For this purpose the grading of the primary group and the grading of the secondary group can be regarded as one total grading with $(n_1+n_2)$ trunks and the availability $(k_1+k_2)$.

The loss $B_2$ in the secondary group is then found easily as

$$ B_2 = \frac{B_{tot}}{B_1} \quad (1) $$

(From the probabilities of state of the total grading the loss $B_1$ can be calculated as well, so that only one set of equations must be solved.)

For the system shown in fig. 2 and an offered traffic of $A = 8$ Erlangs the following loss probabilities and overflow traffic values are obtained:

$B_1 = 0.3355$  $R_1 = 2.684$ Erlangs

$B_2 = 0.1142$  $R_2 = 0.307$ Erlangs

$B_{tot} = 0.0383$

The number of equations increases very rapidly with the number of trunks $(n_1+n_2)$. (In the
2. OVERFLOW SYSTEMS WITH ONE NON-IDEAL GRADING AND ONE IDEAL GRADING (OR FULL AVAILABLE GROUP)

2.1. Overflow Systems with a Non-ideal Primary Grading and an Ideal Secondary Grading

2.1.1. The System

This section deals with overflow systems consisting of a non-ideal primary grading and an ideal secondary grading as shown in fig. 3.

Fig. 3: Non-ideal primary grading with ideal secondary grading

For the following calculation method, equal offered overflow traffic to each secondary selector group is presumed. This condition is fulfilled if the uniform number of primary and secondary selector groups is determined as follows: Let \( g'_1 \) be the original number of primary selector groups and \( g'_2 \) that of the secondary selector groups. Then the uniform number of selector groups in the overflow system has to be

\[
g = g'_1 \cdot g'_2
\]

where

\[
g'_2 = (g'_2)
\]

(2)

Primary and secondary selector groups have to be arranged such that each combination of a certain primary selector group and a certain secondary selector group occurs just once, as indicated schematically in fig. 3.

2.1.1.2. The Equations of State

The trunks in the primary group must be numbered (in an arbitrary order). Then the grading of this group can be described by means of a matrix \( M \) with the general element \( m_{s,j} \) (\( s=1..k_1, j=1..g'_1 \)). The matrix corresponding to the primary group in fig. 3 is shown in fig. 4.

Fig. 4: The matrix \( M \)

As the state congestion probabilities

\[
G_2(x_2) = \begin{pmatrix} \frac{2}{x_2} \\ \frac{2}{x_2} \end{pmatrix}, \quad x_2 = 0, 1, ..., n_2
\]

(7)

of the ideal secondary grading and the corresponding passage probabilities

\[
\mu_2(x_1) = -\frac{G_2(x_2)}{x_2}, \quad x_2 = 0, 1, ..., n_2
\]

(8)

are known, it is not necessary to regard all its possible \( 2^n \) patterns of established calls; it is sufficient to consider the \( (2^n + 1) \) different global states.

For the description of the states "free" and "busy" of each individual trunk in the primary group a set of Boolean variables \( z_1 (1=1..n_1) \) with the following definition is used:

\[
z_1 = 0 \quad \text{if trunk No. 1 is free,}
\]

\[
z_1 = 1 \quad \text{if trunk No. 1 is busy.}
\]

The probability that the lines No. 1, 2, \( \ldots \), \( n_1 \) of the primary group have a certain state \( \{z_1, z_2, \ldots, z_{n_1}\} \) and that, furthermore, just \( x_2 \) trunks are busy in the secondary group is denoted by \( p(z_1, z_2, \ldots, z_{n_1}; x_2) \). Then the following equations of state are obtained according to the principle of statistical equilibrium:

\[
\left[ \sum_{j=1}^{n_1} z_j \cdot 2^x \right] \cdot \left( \frac{1}{g'_1} \cdot \frac{g'_2}{g_2} \right) \cdot \prod_{j=1}^{n_1} \prod m_{s,j} \cdot p(z_1, z_2, \ldots, z_{n_1}; x_2)
\]

(5a)

\[
\frac{\sum_{j=1}^{n_1} z_j \cdot 2^x \cdot \mu_2(x_1)}{g'_1} \cdot \frac{g'_2}{g_2} \cdot \prod m_{s,j} \cdot p(z_1, z_2, \ldots, z_{n_1}; x_2)
\]

(5b)

where

\[
\alpha = 1, \quad \beta = 0
\]

\[
\alpha = 0, \quad \beta = 0
\]

\[
\beta = 0, \quad \beta = \frac{x_2}{x_2} \quad \text{if } x_2 > 0
\]

The product \( \prod m_{s,j} \) in the fourth row of equation (5a) refers to the first \( x_1 \) outlets of a certain selector group. The summation \( \sum_{j=1}^{n_1} \) comprises all busy-patterns of this selector group (column of the matrix \( M \)) where at least the first \( x_1 \) outlets are busy.

The sum of all probabilities \( p(z_1, z_2, \ldots, z_{n_1}; x_2) \) is equal to one:

\[
\sum_{j=1}^{n_1} \sum_{j=1}^{n_1} \sum_{j=1}^{n_1} \cdot 2^x \cdot \mu_2(x_1) = 1
\]

(6)

For the solution of the equations (5a,b) theSOR-method is suitable.

2.1.2. The Loss Probabilities

The overflow traffic \( R_2 \) (see fig. 3) can easily be obtained from the state probabilities:

\[
R_2 = \frac{g'_2}{g_2} \sum_{j=1}^{n_1} \sum_{j=1}^{n_1} \sum_{j=1}^{n_1} \cdot 2^x \cdot p(z_1, z_2, \ldots, z_{n_1}; x_2) \prod m_{s,j}
\]

(6)

Analogously the overflow traffic \( R_1 \) amounts to

\[
R_1 = \frac{g'_2}{g_2} \sum_{j=1}^{n_1} \sum_{j=1}^{n_1} \sum_{j=1}^{n_1} \cdot 2^x \cdot p(z_1, z_2, \ldots, z_{n_1}; x_2) \prod m_{s,j}
\]

(7)

With these values and with the offered traffic \( A \) one can calculate easily the loss probability \( R_2 \) of the primary group, the loss \( R_2 \) of the secondary group and the total loss \( R_2 \):
Example

For a primary grading, as shown in fig. 3, and an ideal secondary grading with \( n_2 = 10 \), \( k_2 = 4 \) the following values are obtained if a traffic of \( A = 8 \) Erlangs is offered:

- \( B_1 = 0.2108 \) Erlangs
- \( R_1 = 1.6862 \) Erlangs
- \( B_2 = 0.0103 \) Erlangs
- \( R_2 = 0.00173 \) Erlangs
- \( B_{\text{tot}} = 0.00216 \)

The number of selector groups according to eq. (2) and (3) is

\[ g = 4 \cdot (10) = 840 \]

Overflow Systems with a Non-ideal Primary Grading and a Full Available Secondary Group

Systems of this kind represent the special case \( k_2 = n_2 \) of the systems considered in section 2.1 (see fig. 5). Regarding that here

\[ G(x_2) = \begin{cases} 0 & \text{for } x_2 < n_2 \\ 1 & \text{for } x_2 = n_2 \end{cases} \]

the equations of state can be slightly simplified.

Example

For a primary grading as shown in fig. 5 with an offered traffic of \( A = 8 \) Erlangs and a full available secondary group of 10 trunks the values

- \( B_2 = 0.001778 \) Erlangs
- \( B_{\text{tot}} = 0.000291 \)

are obtained.

Overflow Systems with an Ideal Primary Grading and a Non-ideal Secondary Grading

In such an overflow system (as shown in fig. 6), the primary group has the state congestion probabilities

\[ G_1(x_1) = \frac{\binom{x_1}{n_1}}{\binom{n_1}{k_1}}, \quad x_1 = n_1, \ldots, n_1 \]

and the passage probabilities

\[ M(x_2) = \frac{1}{n_1 + 1}, \quad x_2 = 0, 1, \ldots, n_2 \]

As explained in section 2.1.1 the total number of selector groups has to be

\[ g = \frac{G_1^1 \cdot G_2^2}{(k_1)} \]

In this case we get

\[ g = \frac{G_1^1 \cdot G_2^2}{(k_1)} \]

Upon numbering the trunks of the secondary group, the states of these trunks and the matrix of the grading can be denoted as in section 2.1.1. Let the probability that just \( x_1 \) trunks are busy in the ideal primary grading and that furthermore the trunks No. 1, 2, \ldots, \( n_2 \) of the secondary group are in a certain state \( \{z_1, z_2, \ldots, z_{n_2}\} \) be denoted as \( p(x_1; z_1, z_2, \ldots, z_{n_2}) \). Then the following equations of state are obtained:

\[
\frac{d}{dt} p(x_1; z_1, z_2, \ldots, z_{n_2}) = \alpha (1-\beta) p(x_1; z_1, z_2, \ldots, z_{n_2}) \\
+ \sum_{j=1}^{n_2} \sum_{i=1}^{n_2} \sum_{l=1}^{n_1} \sum_{m=1}^{n_1} \frac{\binom{n_2}{n_1}}{n_2} p(0; z_1, z_2, \ldots, z_{n_2}) p(x_1; x_1, x_1, \ldots, x_1) \\
+ \sum_{j=1}^{n_2} \sum_{i=1}^{n_2} \sum_{l=1}^{n_1} \sum_{m=1}^{n_1} \frac{\binom{n_2}{n_1}}{n_2} p(x_1; x_1, x_1, \ldots, x_1) p(0; z_1, z_2, \ldots, z_{n_2}) \]

where

- \( \alpha = 1 \) if \( x_1 = n_1 \), else \( \alpha = 0 \)
- \( \beta = 1 \) if \( x_1 = 0 \), else \( \beta = 0 \)

with the normalizing condition

\[ \sum_{x_1} \sum_{z_1} \sum_{z_2} \sum_{z_3} \sum_{z_4} \sum_{z_{n_2}} p(x_1; z_1, z_2, \ldots, z_{n_2}) = 1 \]

The SOB-method is suitable for solving these eq. (14a,b). From the probabilities \( p(x_1; z_1, z_2, \ldots, z_{n_2}) \) the overflow traffic \( R_2 \) is obtained:

\[ R_2 = \frac{\sum_{x_1} \sum_{z_1} \sum_{z_2} \sum_{z_3} \sum_{z_4} \sum_{z_{n_2}} G_1(x_1) G_2(z_2, \ldots, z_{n_2})}{\sum_{x_1} \sum_{z_1} \sum_{z_2} \sum_{z_3} \sum_{z_4} \sum_{z_{n_2}} G_1(x_1) G_2(z_2, \ldots, z_{n_2})} \]

The overflow traffic \( R_2 \) (and the loss \( B_1 \), resp.) can be determined according to Erlang's interconnection formula. Then the loss probabilities \( B_2 \) and \( B_{\text{tot}} \) can be found with eq. (9) and (10).

Example

For an ideal primary grading with \( n_1 = 10 \), \( k_1 = 4 \) and a secondary grading as shown in fig. 6 one obtains the following values:

- \( B_1 = 0.19938 \) Erlangs
- \( B_2 = 0.001079 \) Erlangs
- \( B_{\text{tot}} = 0.000215 \)
2.4. Overflow Systems with a Full Available Primary Group and a Non-Ideal Secondary Grading

This type of system is a special case of the systems considered in section 2.3. Since in this special case
\[ G_1(x_1) = \begin{cases} 0 & \text{for } x_1 \neq n_1 \\ 1 & \text{for } x_1 = n_1 \end{cases} \]  
(16)
the equations of state (14a) can be simplified. This kind of overflow system is realized very often in telephone networks with small (and therefore full available) primary groups and large final groups with limited access. The investigation of such systems enables detailed studies about the effects of various grading structures on the loss probability in the case of offered overflow traffic.

Example.

Let a random traffic of \( A = 8 \) Erlangs be offered to a full available primary group with \( n_1 = 10 \) trunks and a secondary grading as shown in fig. 6. Then the following loss and overflow traffic values are obtained:

\[
\begin{align*}
B_1 &= 0.12166 \quad R_1 = 0.973 \quad \text{Erlangs} \\
B_2 &= 0.00992 \quad R_2 = 0.00965 \quad \text{Erlangs} \\
B_{\text{tot}} &= 0.00121
\end{align*}
\]

3. OVERFLOW SYSTEMS WITH IDEAL GRADINGS AND FULL AVAILABLE GROUPS

3.1. Overflow Systems with Two Ideal Gradings

Such systems (as shown in fig. 7) can be calculated exactly by means of Bretschneider's method [4]. For comparison, however, the equations of state shall be mentioned briefly:

\[
\begin{align*}
[&x_1 + x_2 + A(1 - \delta(x_1, n_1 \cdot \delta(x_2))] \cdot p(x_1, x_2) \\
&= \alpha(x_1+1) \cdot p(x_1+1, x_2) \\
&+ \beta(x_2+1) \cdot p(x_1, x_2+1) \\
&+ A \cdot \mu_1(x_1+1) \cdot p(x_1+1, x_2) \\
&+ A \cdot \delta(x_1) \cdot [\mu_2(x_2+1) \cdot p(x_1, x_2+1)]
\end{align*}
\]  
(17a)

where

\[
\begin{align*}
\alpha &= 1 \quad \text{if } x_1 < n_1 \quad \text{else } \alpha = 0 \\
\beta &= 1 \quad \text{if } x_2 < n_2 \quad \text{else } \beta = 0 \\
\gamma &= 1 \quad \text{if } x_1 > 0 \quad \text{else } \gamma = 0 \\
\delta &= 1 \quad \text{if } x_2 > 0 \quad \text{else } \delta = 0
\end{align*}
\]

with the normalizing condition

\[
\sum_{x_1=0}^{n_1} \sum_{x_2=0}^{n_2} p(x_1, x_2) = 1
\]  
(17b)

Here \( p(x_1, x_2) \) means the probability that \( x_1 \) trunks are busy in the primary group and \( x_2 \) in the secondary group. The equations (17a,b) can be solved with the aid of the SOH-method.

Example.

For \( n_1 = 10, n_2 = 4, B_1 = 10, K_2 = 4, \) and an offered traffic of \( A = 8 \) Erlangs one obtains

\[
\begin{align*}
B_2 &= 0.009906 \quad R_2 = 0.01516 \quad \text{Erlangs} \\
B_{\text{tot}} &= 0.001205
\end{align*}
\]

3.2. Overflow Systems with an Ideal Primary Grading and a Full Available Secondary Group

Overflow systems of this kind are a special case of the systems mentioned in section 3.1. Regarding, however, eq. (11), the formulae mentioned in section 3.1 can be slightly simplified.

Example.

For \( n_1 = 10, n_2 = 4, B_2 = 10, \) and an offered traffic of \( A = 8 \) Erlangs one obtains the values

\[
\begin{align*}
B_2 &= 0.001278 \quad R_2 = 0.00204 \quad \text{Erlangs} \\
B_{\text{tot}} &= 0.000255
\end{align*}
\]

3.3. Overflow Systems with a Full Available Primary Group and an Ideal Secondary Grading

3.3.1. The System and the Equations of State

Such systems (as shown in fig. 8) could be calculated according to the method mentioned in section 3.1. If, however, eq. (16) is taken into account, the equations of state can be simplified remarkably, and a new type of equation is obtained:

\[
\begin{align*}
[&x_1 + x_2 + A(1 - \delta(x_1, n_1 \cdot \delta(x_2))] \cdot p(x_1, x_2) \\
&= (x_1+1) \cdot p(x_1+1, x_2) \\
&+ (x_2+1) \cdot p(x_1, x_2+1) \\
&+ A \cdot \mu_1(x_1+1) \cdot p(x_1+1, x_2) \\
&+ A \cdot \mu_2(x_2+1) \cdot p(x_1, x_2+1)
\end{align*}
\]  
(18a)

and with the normalizing condition

\[
\sum_{x_1=0}^{n_1} \sum_{x_2=0}^{n_2} p(x_1, x_2) = 1
\]  
(18b)

with

\[
p(x_1, x_2) = 0 \quad \text{for } x_1 < 0, \\
x_1 > n_1, \\
x_2 < 0, \\
x_2 > n_2
\]  
(18c)

It should be pointed out that in (18a) there is no term corresponding to the last term in eq. (17a).

In the following, an explicit solution of the equations (18a,b,c) will be derived.
### 1.3.2. Graphic Illustration of the Equations of State

In the following schemes (fig. 9, 10, 11, 12) each of the probabilities \( p(x_1,x_2) \) is represented by a crosspoint of a grid as shown in fig. 9.

![Grid Representation](image)

**Fig. 9:** Representation of the probabilities \( p(x_1,x_2) \) by the crosspoints of a grid.

The eq. (18a,b) connect 2, 3, or 4 probabilities \( p(x_1,x_2) \) each. These equations are represented in fig. 10, 11, 12 by small graphs which connect the points corresponding to the p-values.

**Fig. 10:** Illustration of eq. (18a) for \( x_1 = 4, x_2 = 3 \). In fig. 11 some more examples of eq. (18a) are indicated, including the special cases \( x_1 = 0, x_2 = n_2 \). Fig. 12 shows examples corresponding to eq. (18b).

![Graphs](image)

**Fig. 11:** Illustration of eq. (18a) for \( x_1 = 4, x_2 = 3 \).

**Fig. 12:** Illustration of eq. (18b).

### 2.3.3. Reduction to a One-dimensional System

In a first step, the probabilities \( p(x_1,x_2) \) can be expressed as a function of the values \( p(0,x_2) \) only.

For \( x_1 = 0 \) and \( x_2 = n_2 \), eq. (18a) yields a relation between \( p(0,n_2) \) and \( p(1,n_2) \) as indicated in the upper left corner of fig. 11. Thus, if the probability \( p(0,n_2) \) is given, also \( p(1,n_2) \) can be calculated.

For \( x_1 = 1 \) and \( x_2 = n_2 \), eq. (18a) constitutes a relation between \( p(0,n_2) \), \( p(1,n_2) \), and \( p(2,n_2) \). Therefore, knowing \( p(0,n_2) \) and \( p(1,n_2) \), the pr. \( p(2,n_2) \) can be determined, and so forth.

Obviously all probabilities \( p(x_1,x_2) \) are functions of the pr. \( p(0,n_2) \) only.

For \( x_1 = 0 \), \( x_2 = n_2 - 1 \) eq. (18a) contains the pr. \( p(0,n_2) \), \( p(0,n_2-1) \), and \( p(1,n_2-1) \). Thus, if also \( p(0,n_2-1) \), besides \( p(0,n_2) \), is given, the pr. \( p(1,n_2-1) \) can be evaluated. With eq. (18a) for \( x_1 = 1 \) and \( x_2 = n_2 - 1 \) the pr. \( p(2,n_2-1) \) can be obtained, etc.

Thus, all pr. \( p(x_1,n_2) \) can be expressed as a function of only the \( (n_2+1) \) probabilities \( p(0,0), p(0,1), \ldots, p(0,n_2) \), which are situated at the left edge of the grid:

\[
\rho(x_1,x_2) = f_{x_1,x_2}(p(0,0), p(0,1), \ldots, p(0,n_2))
\]  

(19)

If these functions \( f_{x_1,x_2} \) according to eq. (19) are calculated along the method shown above, the following expression is found:

\[
P(X_1,X_2) = \sum_{x_1} (-1)^{x_1-1} \cdot \frac{S_{x_2}}{S_{x_1}} \cdot P(O, x_1) \cdot f_{x_1,x_2}(p(0,0), p(0,1), \ldots, p(0,n_2))
\]  

(20a)

where

\[
S_{x_1} = \sum_{x_2} \frac{A^{m-v}}{(m-v)!} \cdot \frac{S_{x_2}}{S_{x_1}}
\]  

(20b)

The formula (20a,b) holds true also in the special case of a full available secondary group (instead of an ideal second grading considered here). In connection with the calculation of such overflow systems with full available secondary groups (see section 3.4), formula (20a,b) has already been derived by E. Brockmeyer [4].

The same formula is obtained in the more general case considered here, because the state congestion probabilities \( \rho(x_2) \) of the secondary system, which make the difference between these systems, do not occur in eq. (18a) and (20a,b).

### 2.3.4. Reduction to the Probability \( p(0,n_2) \)

In this section it will be shown that the pr. \( p(0,n_2) \) can be expressed by the pr. \( p(0,n_2) \) with the aid of eq. (18b):

Inserting eq. (20a) into (18b), one obtains

\[
A \cdot \mu(x_2) \cdot \sum_{x_1} (-1)^{x_1} \cdot \frac{S_{x_1}}{S_{x_2}} \cdot P(O, x_1) = [A \cdot \mu(x_2) + n_1 + x_1] \cdot \sum_{x_2} (-1)^{x_2} \cdot \frac{S_{x_2}}{S_{x_1}} \cdot P(O, x_2) - A \cdot \sum_{x_2} (-1)^{x_2} \cdot \frac{S_{x_2}}{S_{x_1}} \cdot P(O, x_2)
\]  

(21)

or, substituting \( x_2 \) by \( (x_2+1) \),

\[
P(O,n_2) = A \cdot \mu(x_2) \cdot \sum_{x_1} (-1)^{x_1} \cdot \frac{S_{x_1}}{S_{x_2}} \cdot P(O, x_1)
\]  

(22)
From the eq. (18a) and (20a) follows for \( x_2 = n_2 \\
( A+x_1)n_2 \cdot S_{n_2} x_1 + A \cdot S_{n_2} x_1 \\
or, generally \((x_1 \mapsto m, n_2 \mapsto r): \\
( A+m+r) \cdot S_{r,m} = (m+r) \cdot S_{r,m} + A \cdot S_{r,m-1} \)
Inserting eq. (23) into (22) for \( r \mapsto \xi \) and \( m \mapsto n_1 x_2 + 1 - E \), and regarding that \\
\((x_2+1) \cdot (x_2+2) = (x_2+1) \cdot (x_2+2) \)
one obtains \\
\( p(o,x_2) = \frac{1}{S_{n_1 x_2}} \sum_{x_2} (-1)^{x_2-1} \cdot p(0,\xi) \).

Using eq. (24) or, more conveniently, from the equation \\
\( \sum_{x_2} p(o,x_2) = \frac{A^2}{A} \sum_{x_2} \frac{x_2}{A} = \frac{S_{o,n_2}}{S_{o,n_2}} \)
Using eq. (31) for \( x_2 = 0 \), one obtains \\
\( \sum_{x_2} \frac{p(o,x_2)}{A^{x_2}} = \frac{A^{n_2}}{S_{o,n_2}} \) 

In setting eq. (28b) yields \\
\( p(o,n_2) \cdot \sum_{x_2} \frac{b_{x_2}}{A^{x_2}} = \frac{A^{n_2}}{S_{o,n_2}} \cdot p(o,n_2) \)

From the eq. (30) and (34) the solution for the state probabilities \( P(x_1,x_2) \) is obtained: \\
\( \sum_{x_1} p(x_1,x_2) = \frac{A^{n_2}}{S_{n_1 x_2}} \cdot \frac{n_2}{A^{x_2}} \cdot b_{x_2} \) 

The overflow traffic \( R_2 \) amounts to \\
\( R_2 = A \sum_{x_2} \frac{\alpha(x_2) \cdot p(n_2,x_2)}{x_2} \)

The loss \( B_1 \) (or the overflow traffic \( R_1 \), resp.) can be calculated according to Erlang's loss formula, the loss values \( B_2 \) and \( B_{\text{tot}} \) result from eq. (9) and (10).

3.3.6. Numerical Computation and Results

Using the relation \\
\( S_{r,m} = S_{r+1,m} + S_{r,m-1} \)
the S-polynomials can be evaluated successively by addition of two other values, starting with the numerical values \\
\( S_{r,0} = 1 \) and \( S_{0,m} = \frac{A^m}{m!} \)

Example. 
For an overflow system with a full available primary group of \( n_1 = 10 \) trunks, an offered traffic of \( A = 8 \) Erlangs, and an ideal secondary grading of \( n_2 = 10 \) trunks and availability \( k_2 = 4 \) one obtains \\
\( B_2 = 0.008889 \quad B_{\text{tot}} = 0.008625 \) Erlangs \\
\( S_{0,4} = 0.001081 \)

or calculating this example, the method shown here is faster by a factor 35 as compared with the SOR method.

3.3.7. Extension to Non-ideal Secondary Gradings

This method can also be applied to overflow systems with non-ideal secondary gradings (see section 2.4) if the state congestion probabilities \( \Phi_2(x_2) \) are known with suf-
cient accuracy. For determining $C_2(x_2)$, the following approximation formula by "U." Herzog and R. Kirsch[6] can be used:

$$C_2(x_2) = \binom{x_2}{k_2} \cdot \frac{n_2 - k_2}{k_2} \cdot \frac{\gamma_2}{\sqrt{k_2 - 6}} \cdot \left(\frac{k_2}{k_2 - 6}\right)^{x_2},$$

with

$$k^* = k_2 - c_1 \cdot (k_2 - 5) \cdot \frac{n_2 - k_2}{k_2} \cdot \frac{\gamma_2}{\sqrt{k_2 - 6}},$$

where $c_1$ and $c_2$ are constants which can be found by a few simulation runs per grading type.

Example.

For an overflow system with $n_1 = 36$ trunks and a secondary group which consists of a so-called simplified standard grading (as shown in fig. 13)

with $n_2 = 60$ trunks and the availability $k_2 = 10$

one obtains for an offered traffic of $A = 80$ Erlangs the value $B_2 = 0.0752$

(For so-called simplified standard gradings of the German PTT the constants $c_1 = 1.26$ and $c_2 = 0.32$ are valid [6]. The traffic $y_2$ carried in the secondary group was determined approximately for offered overflow traffic.)

A traffic simulation yields

$$B_2 = 0.0750 \pm 0.004$$

From this it can be seen that by means of this numerical method loss probabilities of high accuracy can be obtained even for remarkably large secondary groups.

3.4. Overflow Systems Consisting of Two Full Available Groups

The loss values in overflow systems where the primary group as well as the secondary group are full available can be easily obtained by applying Erlang's loss formula two times. The probability distribution in the secondary group can be calculated according to the well known analytic solution of E. Brockmeyer [5].

5. Overflow Systems with Two Full Available Groups and a Finite Number of Sources

An overflow system consisting of 2 full available groups with a limited number of traffic sources is shown in fig. 14. For this system the following equations of state are obtained:

$$[x_1 + x_2 + \alpha(q - x_1 - x_2)]p(x_1, x_2) = \alpha(q - x_1 - x_2)p(x_1, x_2) + (x_1 + 1)p(x_1, x_2) + (x_2 + 1)p(x_1, x_2 + 1)$$

where $\alpha = 1$ if $x_2 < n_2$, else $\alpha = 0$.

$$\sum_{x_1=0}^{n_1} \sum_{x_2=0}^{n_2} p(x_1, x_2) = 1$$

An analytic solution has been derived also for this type of overflow systems. For lack of space, however, this solution will be only briefly stated here.

The following expression for the probabilities $p(x_1, x_2)$, which is similar to eq. (19), can be derived with the aid of generating functions:

$$p(x_1, x_2) = \frac{\gamma_1}{x_1} \cdot \sum_{x_1=0}^{n_1} \sum_{x_2=0}^{n_2} \left(\begin{array}{c} x_1 \\ x_2 \end{array}\right) \cdot \left(\begin{array}{c} x_1 \\ x_2 \end{array}\right) \cdot \frac{\gamma_1}{x_1} \cdot \frac{\gamma_2}{x_2} \cdot \frac{\gamma_2}{x_2} \cdot \frac{\gamma_2}{x_2} \cdot \frac{\gamma_2}{x_2} \cdot \frac{\gamma_2}{x_2} \cdot \frac{\gamma_2}{x_2} \cdot \frac{\gamma_2}{x_2}$$

where

$$T_{r,m} = \sum_{q=0}^{\infty} \left(\begin{array}{c} r \\ m \end{array}\right) \cdot \sum_{q=0}^{\infty} \left(\begin{array}{c} q \\ q \end{array}\right) \cdot \frac{q}{m} \cdot \frac{q}{m} \cdot \frac{q}{m} \cdot \frac{q}{m} \cdot \frac{q}{m} \cdot \frac{q}{m} \cdot \frac{q}{m}$$

The value $p(n_1, n_2)$ can be calculated directly:

$$p(n_1, n_2) = \frac{(n_1 + 2) \cdot (n_2 + 2)}{n_1 \cdot n_2}$$

Furthermore, eq. (41) yields

$$p(n_1, n_2) = p(0, 0) \cdot T_{n_1, n_2}$$

thus

$$p(0, 0) = \frac{(n_1 + 2) \cdot (n_2 + 2)}{T_{n_1, n_2} \cdot n_1 \cdot n_2}$$

Inserting eq. (45) into (40b) leads to the relation.
where $\psi = 1$ if $x_2 < n_2 - 1$, else $\psi = 0$, or in a simplified notation

$$p(0, x_2) = \sum_{x_2=0}^{n_2} \sum_{x_1=0}^{n_1} \alpha_{x_2, x_1} \cdot p(0, x_1, x_2)$$  \hspace{1cm} (47)

with the abbreviation

$$\alpha_{x_2, x_1} = \frac{(-\rho)^{x_2}}{\rho}, (n_2 - x_2)! \cdot \frac{n_1}{(n_1 + x_1)!} \cdot \frac{x_2}{x_1}$$

\hspace{1cm} (44)

where $\psi = 1$ if $x < n_2 - 1$, else $\psi = 0$.

Knowing $p(0, x_2)$ from eq. (44), all values $p(0, x_2)$ can be calculated successively according to eq. (47). Then the probabilities $p(x_1, x_2)$ can be easily evaluated by means of eq. (41).

This results in the following formulae for the probabilities $p(x_1, x_2)$:

$$p(x_1, x_2) = \frac{(\rho q)^{x_2}}{x_2!} \cdot \frac{n_1}{(n_1 + x_1)!} \cdot \frac{x_2}{x_1} \cdot \frac{\alpha_{x_2, x_1}}{\rho} \cdot \sum_{x_2=0}^{n_2} \sum_{x_1=0}^{n_1} \alpha_{x_2, x_1}$$

\hspace{1cm} (49)

where $\alpha_{x_2, x_1}$ is defined in eq. (44).

Further one obtains the traffic $Y_1$ carried in the primary group and the traffic $Y_2$ carried in the secondary group:

$$Y_1 = \sum_{x_1=0}^{n_1} \sum_{x_2=0}^{n_2} x_1 \cdot p(x_1, x_2)$$  \hspace{1cm} (51)

$$Y_2 = \sum_{x_1=0}^{n_1} \sum_{x_2=0}^{n_2} x_2 \cdot p(x_1, x_2)$$  \hspace{1cm} (52)

the offered traffic

$$A = \alpha \cdot (q - Y_1 - Y_2)$$  \hspace{1cm} (53)

and the overflow traffics

$$R_1 = R_1 - Y_1$$  \hspace{1cm} (55)

$$R_2 = R_2 - Y_2$$

The loss values $B_1$, $B_2$, and $B_{tot}$ result from eq. (8), (9), and (10).

For the variance $V$ of the traffic $Y_2$ carried in the secondary group one obtains

$$V = \sum_{x_1=0}^{n_1} \sum_{x_2=0}^{n_2} (x_2 - Y_2)^2 \cdot p(x_1, x_2)$$  \hspace{1cm} (56)

Example,

For a full available primary group with $n_1 = 10$ trunks, a full available secondary group with $n_2 = 10$ trunks, and for $q = 40$ traffic sources with a call intensity $\alpha = 0.5$, the following values are obtained:

$$Y_1 = 8.748 \text{ Erlangs}$$

$$Y_2 = 4.505 \text{ Erlangs}$$

$$A = 13.373 \text{ Erlangs}$$

The following values are obtained:

$$R_1 = 0.3958 \text{ Erlangs}$$

$$R_2 = 0.120 \text{ Erlangs}$$

LITERATURE


